

OPTIMAL QUOTIENTS OF MUMFORD CURVES AND COMPONENT GROUPS

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1. INTRODUCTION

Let X be a smooth, projective, geometrically irreducible curve defined over a local field K . Assume X has degenerate reduction; see (2.9) for the definition. Let J be the Jacobian variety of X . We say that an elliptic curve E is an *optimal quotient* of X if there is a finite morphism $X \rightarrow E$ defined over K such that the homomorphism $\pi : J \rightarrow E$ induced by the Albanese functoriality has connected and reduced kernel. The following question, originally posed by Ribet and Takahashi, appears in [2]:

1.1. Question. *Is the functorially induced map $\pi_* : \Phi_J \rightarrow \Phi_E$ on component groups of the Néron models of J and E necessarily surjective?*

We will construct examples which show that the answer is No, contrary to the expectation expressed in [2]. The interest in Question 1.1 comes from arithmetic geometry, where for certain modular curves the answer was known to be positive. It is natural then to ask whether the surjectivity of the map on component groups is a general geometric property of curves with degenerate reduction, or is a special arithmetic property of modular curves with degenerate reduction. Our examples indicate that the latter is the case.

A positive answer to Question 1.1 can be deduced for two types of modular curves with degenerate reduction, using various results from arithmetic geometry. The first is the Drinfeld modular curve $X_0(\mathfrak{n})$ of arbitrary level $\mathfrak{n} \in \mathbb{F}_q[T]$ considered over $\mathbb{F}_q((1/T))$. The proof that π_* is surjective in this case relies on the interpretation of the uniformizing lattice of the Jacobian of $X_0(\mathfrak{n})$ as a space of automorphic forms on $\mathrm{GL}(2)$; see (4.26). The second is the modular curve $X_0(p)$ of prime level p considered over \mathbb{Q}_p . Here the surjectivity of π_* follows from some deep results of Mazur and Ribet; see (4.27). There is a third family of modular curves with degenerate reduction. Let $D > 1$ be a square-free integer divisible by an even number of primes, and $M \geq 1$ be an integer coprime to D . Let $X_0^D(M)$ be the Shimura curve over \mathbb{Q} associated with an Eichler order of level M in an indefinite quaternion algebra over \mathbb{Q} of discriminant D . For any $p|D$, $X_0^D(M)$ over \mathbb{Q}_p is a curve with degenerate reduction. Theorem 2.4 in [32] claims that π_* is surjective in this case. The proof of this theorem crucially relies on a result of Bertolini and Darmon [3, Prop. 4.4]. Unfortunately, the proof of this latter Proposition has a gap, cf. (4.28), so Question 1.1 in this case remains a very interesting open problem.

In our examples giving a negative answer to Question 1.1 the curve X has genus two. The study of curves whose Jacobians are isogenous to a product of two elliptic

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curves has a long history, dating back to Legendre and Jacobi. In more recent times such curves have found applications in a variety of arithmetic problems, for example, construction of curves with a maximal number of rational points over finite fields [31], or construction of Jacobians over \mathbb{Q} with large rational torsion subgroups [18].

The idea of our construction, which is discussed in Section 3, is to take two elliptic curves E_1 and E_2 over K with non-trivial component groups, and to consider a quotient of the abelian surface $E_1 \times E_2$ by a finite subgroup-scheme G . We show that G can be chosen so that the resulting quotient J is a Jacobian variety with toric reduction and $\Phi_J = 1$. Moreover, E_1 and E_2 are optimal quotients of J . Clearly the corresponding maps on component groups cannot be surjective.

From the work of Mumford and others it is known that abelian varieties with toric reduction have rigid-analytic uniformization. In Section 4, we investigate the map $\pi_* : \Phi_J \rightarrow \Phi_E$ using analytic techniques. We show that the Tate period of E can be obtained from J via a natural evaluation map. In this construction, which is a generalization of the constructions due to Gekeler and Reversat [15], Bertolini and Darmon [3], and Takahashi [33], the uniformizing lattice of J maps to a subgroup in K^\times isomorphic to $\mathbb{Z}/c\mathbb{Z} \oplus \mathbb{Z}$. We show that the cokernel of π_* is isomorphic to $\mathbb{Z}/c\mathbb{Z}$. We also show that $\#\text{coker}(\pi_*)$ is closely related to the denominator of the idempotent in $\text{End}(J) \otimes \mathbb{Q}$ corresponding to E , and to the ratio of the degree of the morphism $X \rightarrow E$ and the congruence number of E . These results are of independent interest, and could be useful in the theory of Mumford curves and for a computational investigation of Question 1.1 in the case of Shimura curves: see (4.30). The main theorem of this section is Theorem 4.21, which gives equivalent conditions for π_* to be surjective. We then explain why these conditions hold for certain modular curves.

In Section 5, we reexamine (and slightly generalize) the construction of Section 3 using analytic methods, thus illustrating the results of Section 4.

2. NÉRON MODELS

For the convenience of the reader and future reference we collect in this section some facts about abelian varieties and their Néron models.

2.1. From now on, K will be a field equipped with a nontrivial discrete valuation

$$\text{ord}_K : K \rightarrow \mathbb{Z} \cup \{+\infty\}.$$

Let $R = \{z \in K \mid \text{ord}_K(z) \geq 0\}$ be its ring of integers. Let $\mathfrak{m} = \{z \in K \mid \text{ord}_K(z) > 0\}$ be the maximal ideal of R , and $k = R/\mathfrak{m}$ be the residue field. We fix a uniformizer ϖ of R , and assume that the valuation is normalized by $\text{ord}_K(\varpi) = 1$. Assume further that k is a finite field of characteristic p , and define the non-archimedean absolute value on K by $|x| = (\#k)^{-\text{ord}_K(x)}$. Finally, assume K is complete for the topology defined by this absolute value. Overall, our assumptions mean that K is a *local field*. It is known that every local field is isomorphic either to a finite extension of \mathbb{Q}_p , or to the field of formal Laurent series $k((x))$. We denote by \mathbb{C}_K the completion of an algebraic closure \bar{K} of K with respect to the extension of the absolute value (which is itself algebraically closed).

2.2. Let X be a scheme over K . A *model* of X over R is an R -scheme \mathcal{X} such that $\mathcal{X}_K = X$. Let A be an abelian variety over K . There is a model \mathcal{A} of A which is smooth, separated, and of finite type over R , and which satisfies the

following universal property: For each smooth R -scheme \mathcal{X} and each K -morphism $\phi_K : \mathcal{X}_K \rightarrow A$ there is a unique R -morphism $\phi : \mathcal{X} \rightarrow \mathcal{A}$ extending ϕ_K ; see [5]. The model \mathcal{A} is called the *Néron model* of A . It is obvious from the universal property that \mathcal{A} is uniquely determined by A , up to unique isomorphism. Moreover, the group scheme structure of A uniquely extends to a commutative R -group scheme structure on \mathcal{A} , and $A(K) = \mathcal{A}(R)$.

2.3. The closed fibre \mathcal{A}_k is usually not connected. Let \mathcal{A}_k^0 be the connected component of the identity section. There is an exact sequence

$$0 \rightarrow \mathcal{A}_k^0 \rightarrow \mathcal{A}_k \rightarrow \Phi_A \rightarrow 0,$$

where Φ_A is a finite étale group scheme over k . The group Φ_A is called the *group of connected components* of A .

2.4. Let $f_K : A \rightarrow B$ be a morphism of abelian varieties. By the Néron mapping property, the morphism f_K extends to a homomorphism $f : \mathcal{A} \rightarrow \mathcal{B}$. Restricting to the closed fibres we get a homomorphism $f_k : \mathcal{A}_k \rightarrow \mathcal{B}_k$. This homomorphism maps \mathcal{A}_k^0 into \mathcal{B}_k^0 . Hence there are induced homomorphisms $f_k^0 : \mathcal{A}_k^0 \rightarrow \mathcal{B}_k^0$ and $f_* : \Phi_A \rightarrow \Phi_B$.

2.5. Let K' be an unramified extension of K . Let R' be the ring of integers of K' . Let $f_{K'} : A_{K'} \rightarrow B_{K'}$ be the base change of f_K to K' . Then $f \otimes R' : \mathcal{A} \otimes_R R' \rightarrow \mathcal{B} \otimes_R R'$ is the corresponding morphism of the Néron models; see [5, Cor. 7.2/2]. This implies that $f_* : \Phi_A(\bar{k}) \rightarrow \Phi_B(\bar{k})$ does not change under unramified field extensions of K .

2.6. By a theorem of Chevalley, \mathcal{A}_k^0 is uniquely an extension of an abelian variety B by a connected affine group $T \times U$, where T is a torus and U is a unipotent algebraic group; see [5, §9.2]. We say that A has *toric reduction* if U and B are trivial. We say that A has *split toric reduction* if it has toric reduction and $T \cong \mathbb{G}_{m,k}^g$ is a split torus.

2.7. If A has toric reduction, and $f_K : A \rightarrow B$ is an isogeny, then f_k^0 is an isogeny; cf. [5, Cor. 7.3/7]. This implies that B also has toric reduction. If $f_K : B \rightarrow A$ is a closed immersion of abelian varieties and A has toric reduction, then f_k^0 is a closed immersion; see the proof of Theorem 8.2 in [8]. This implies that if A has (split) toric reduction, then any abelian subvariety of A also has (split) toric reduction.

2.8. Let X be a smooth, projective, geometrically irreducible curve over K of genus $g \geq 1$. A model \mathcal{X} of X over R is said to be *semi-stable* if \mathcal{X} is flat and proper, and the closed fibre \mathcal{X}_k is reduced and has only ordinary double points as singularities. If X has a semi-stable model \mathcal{X} over R , then \mathcal{X} can be chosen to be regular. Every curve has a semi-stable model after possibly passing to a finite extension of K ; see [5, Thm. 9.2/7].

2.9. We say that X has *degenerate reduction* if X has a semi-stable model \mathcal{X} over R such that the normalizations of all irreducible components of \mathcal{X}_k are isomorphic to \mathbb{P}_k^1 . We say that X has *split degenerate reduction* if it has degenerate reduction and all double points of \mathcal{X}_k are k -rational with two k -rational branches; such curves are also called *Mumford curves*.

2.10. Let $J = \text{Pic}_{X/K}^0$ be the Jacobian variety of X . Let \mathcal{J} be the Néron model of J over R . The curve X has (split) degenerate reduction if and only if J has

(split) toric reduction. This follows from the proof of Theorem 2.4 in [9]. The connection between the reductions of X and J is based on Raynaud's isomorphism $\mathrm{Pic}_{\mathcal{X}/R}^0 \xrightarrow{\sim} \mathcal{J}^0$.

2.11. An *optimal quotient* of A is an abelian variety B and a smooth surjective morphism $\pi : A \rightarrow B$ whose kernel is connected (i.e., an abelian variety); cf. [8, Def. 3.1]. Let $\pi : A \rightarrow B$ be an optimal quotient. Denote by A^\vee and B^\vee the abelian varieties dual to A and B , respectively. Then π is an optimal quotient if and only if the dual morphism $\pi^\vee : B^\vee \rightarrow A^\vee$ is a closed immersion; cf. [8, Prop. 3.3].

2.12. We say that an elliptic curve E is an *optimal quotient* of the curve X , if there is a morphism $f : X \rightarrow E$ defined over K which does not factor over \bar{K} as $X \rightarrow E' \rightarrow E$, where E' is an elliptic curve over \bar{K} and $E' \rightarrow E$ is an isogeny with non-trivial kernel; cf. [21].

2.13. Assume X has a K -rational point. Let $f : X \rightarrow E$ be a morphism. We claim that f is an optimal quotient if and only if the homomorphism $\pi : J \rightarrow E$ of Jacobian varieties induced by the Albanese functoriality is an optimal quotient. If f is not optimal, so factors as $X \rightarrow E' \rightarrow E$, then, by functoriality, π also factors through a homomorphism $J \rightarrow E'$. Since $E' \rightarrow E$ has non-trivial kernel, the kernel of π cannot be connected and reduced. Conversely, suppose π is not optimal. Let B be the reduced subscheme underlying the connected component of the identity of $\ker(\pi)$. It is clear that B is an abelian subvariety of J of codimension 1. Hence the quotient $J/B = E'$ is an elliptic curve, and π factors as $J \rightarrow E' \rightarrow E$. Since f is the composition of π with an Abel-Jacobi map $X \hookrightarrow J$, we see that f also factors through E' .

2.14. Suppose X has degenerate reduction. There is a smooth k' -rational point on \mathcal{X}_k for some finite extension k' of k . The geometric version of Hensel's lemma then implies that X has a K' -rational point for some finite unramified extension K' of K . Using (2.5) and (2.13), we conclude that, as far as the questions of surjectivity of component groups are concerned, E being an optimal quotient of J or an optimal quotient of X are equivalent conditions, and we can assume that X is a Mumford curve.

3. JACOBIANS ISOGENOUS TO A PRODUCT OF TWO ELLIPTIC CURVES: ALGEBRAIC APPROACH

We start by giving a very explicit, equation based example. We will explain later in this section how this example can be obtained as a special case of a general construction.

3.1. *Example.* Let $K = \mathbb{Q}_p$, where p is odd. Let X be the hyperelliptic curve of genus 2 defined by the equation

$$y^2 = (px^2 + (p-1))((p+1)x^2 + p)(x^2 + 1).$$

Tate's famous algorithm for computing minimal regular models of elliptic curves over discrete valuation fields has been generalized by Qing Liu to curves of genus two; see [22], [23]. From the equation defining X , with some patience, one computes its Igusa invariants, which are generalizations of the j -invariant of an elliptic curve. Knowing these invariants, we apply Theorem 1 and Proposition 2 in [22] to conclude

that the closed fibre of the minimal regular model of X over R is a projective line intersecting itself transversally in two points and $\Phi_J = 1$. In particular, X has degenerate reduction. Next, let E be the elliptic curve given by the equation $y^2 = x(x-1)(x+p)$. The j -invariant of E is

$$j(E) = 2^8 \frac{(p^2 + p + 1)^3}{p^2(p+1)^2}.$$

Since $\text{ord}_K(j) = -2$, we have $\Phi_E \cong \mathbb{Z}/2\mathbb{Z}$ (by Tate's algorithm). Finally, there is a degree-2 morphism $f : X \rightarrow E$ given by

$$(x, y) \mapsto (p(p+1)x^2 + p^2, p(p+1)y).$$

Clearly f is optimal, but the map $\Phi_J \rightarrow \Phi_E$ cannot be surjective.

3.2. Assume $\#k \neq 2$. Fix an integer $n \geq 2$ dividing $\#k - 1$. Let E_1 and E_2 be two elliptic curves over K with split multiplicative reduction which are not isogenous over the algebraic closure \bar{K} of K . If we denote by q_1 and q_2 the Tate periods of E_1 and E_2 , then this last assumption is equivalent to requiring that $q_1^u \neq q_2^w$ for any non-zero $u, w \in \mathbb{Z}$. Assume $\Phi_{E_1} \cong \Phi_{E_2} \cong \mathbb{Z}/n\mathbb{Z}$ (equiv. $\text{ord}_K(q_1) = \text{ord}_K(q_2) = n$). Then $E_i[n](K) = E_i[n](\bar{K}) \approx \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ ($i = 1, 2$).

3.3. Let

$$e_n : E_i[n] \times E_i[n] \rightarrow \mu_n$$

be the Weil pairing. Recall that the Weil pairing is alternating, i.e., $e_n(P, P) = 1$ for any $P \in E_i[n]$; cf. [20, (2.8.7)]. There is a canonical subgroup of $E_i[n]$ corresponding to $(\mathcal{E}_i^0)_K[n] \cong \mathbb{Z}/n\mathbb{Z}$. Fix a generator g_i of this subgroup, and a generator ζ of μ_n . Since e_n is non-degenerate, we can find $h_i \in E_i[n]$ such that $E_i[n] \approx \langle g_i \rangle \times \langle h_i \rangle$, and $e_n(g_1, h_1) = e_n(g_2, h_2) = \zeta$. Let $\psi : E_1[n] \xrightarrow{\sim} E_2[n]$ be the unique isomorphism such that $\psi(g_1) = h_2$ and $\psi(h_1) = g_2$. This is an anti-isometry with respect to the e_n pairings on $E_1[n]$ and $E_2[n]$ because

$$e_n(\psi(g_1), \psi(h_1)) = e_n(h_2, g_2) = e_n(g_2, h_2)^{-1} = e_n(g_1, h_1)^{-1}.$$

Let $A = E_1 \times E_2$ and let $G \subset A[n]$ be the graph of ψ :

$$G = \{(P, \psi(P)) \mid P \in E_1[n]\}.$$

The product of the canonical principal polarizations on E_1 and E_2 is a principal polarization θ on the product variety $A = E_1 \times E_2$.

3.4. Proposition. *There is a principal polarization on the quotient abelian variety $J := A/G$ defined by G and θ . With this principal polarization, J is isomorphic to the canonically principally polarized Jacobian variety of a smooth projective curve X defined over K . The Jacobian J has toric reduction.*

Proof. The existence of X follows from Theorem 3 in [19]. It is important here that ψ is an anti-isometry, and E_1 and E_2 are not isogenous. The curve X can be defined over K because ψ , by construction, is an isomorphism of Galois modules; cf. [18, Prop. 3]. The claim that J has toric reduction follows from (2.7). \square

3.5. Lemma. $\Phi_J = 1$.

Proof. Clearly $G \subset A(K)$ is a subgroup isomorphic to $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. By (2.2), G extends to a constant étale subgroup-scheme of \mathcal{A} . The restriction to the closed

fibre gives an injection $G \hookrightarrow \mathcal{A}_k(k)$, which composed with $\mathcal{A}_k \rightarrow \Phi_A$ gives a canonical homomorphism $\phi : G \rightarrow \Phi_A$. It is clear that $\Phi_A \cong \Phi_{E_1} \times \Phi_{E_2} \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Since

$$\mathcal{A}_k^0[n] \cong \{(P_1, P_2) \mid P_i \in \langle g_i \rangle\},$$

$G \cap \mathcal{A}_k^0 = 0$. Therefore, ϕ is an isomorphism. Now Theorem 4.3 in [26] implies that

$$\Phi_J \cong \Phi_A/G = 1.$$

□

3.6. Lemma. E_1 and E_2 are optimal quotients of J .

Proof. Note that E_i embeds into J as a closed subvariety since $E_i \cap G = 0$. The claim then follows from (2.11). Alternatively, note that the quotient J/E_1 is isomorphic to $E_2/E_2[n] \cong E_2$, so, by definition, E_2 is an optimal quotient of J . □

3.7. In the special case when $n = 2$, Proposition 4 in [18] allows to compute an explicit equation for X starting with equations for E_1 and E_2 . Moreover, in this case the assumption that E_1 and E_2 are not isogenous can be relaxed to the assumption that E_1 and E_2 are not isomorphic over \bar{K} , i.e., have distinct j -invariants; see [19, Thm. 3]. With this in mind, consider the Legendre curves

$$E_1 : y^2 = x(x-1)(x-p) \quad \text{and} \quad E_2 : y^2 = x(x-1)(x+p)$$

over \mathbb{Q}_p . These curves have distinct j -invariants, split multiplicative reductions, and $\Phi_{E_i} \cong \mathbb{Z}/2\mathbb{Z}$. Let $P_1 = (1, 0)$, $P_2 = (0, 0)$ and $P_3 = (p, 0)$ be the non-trivial elements of $E_1[2]$. Similarly, let $Q_1 = (1, 0)$, $Q_2 = (0, 0)$ and $Q_3 = (-p, 0)$ be the non-trivial elements of $E_2[2]$. Modulo p the point P_1 lies in the smooth locus of \bar{E}_1 , hence its specialization lies in the connected component \mathcal{E}_1^0 of the identity. Define ψ by

$$\psi(O) = O, \quad \psi(P_1) = Q_2, \quad \psi(P_2) = Q_1, \quad \psi(P_3) = Q_3.$$

Using the formulas in [18, Prop. 4], one obtains the equation in Example 3.1.

3.8. Remark. When $n \geq 3$, it seems rather difficult to write down an explicit equation for X . In Section 5 we will compute the p -adic periods of J from the Tate periods of E_1 and E_2 . In [34], Teitelbaum developed a method for computing an equation for a genus 2 curve X with split degenerate reduction from the periods of its Jacobian. Teitelbaum's formulae are p -adic, i.e. the coefficients of the equation of X are given by infinite series.

4. RIGID-ANALYTIC CONSTRUCTIONS

First, we briefly review some facts from the theory of rigid-analytic uniformization of abelian varieties. The ideas in the later part of this section are largely due to Gekeler, Reversat, Ribet, and Zagier; see [14], [15], [28], [35].

4.1. Let $\mathfrak{T} := (\mathbb{G}_{m,K}^g)^{\text{an}}$ be the rigid-analytification of

$$\mathbb{G}_{m,K}^g = \text{Spec} K[Z_1, Z_1^{-1}, \dots, Z_g, Z_g^{-1}].$$

A *character* of \mathfrak{T} is a homomorphism of rigid-analytic groups $\chi : \mathfrak{T} \rightarrow \mathbb{G}_{m,K}^{\text{an}}$. Denote the group of characters of \mathfrak{T} by $\mathcal{X}(\mathfrak{T})$. It is known that analytic characters are all algebraic,

$$\mathcal{X}(\mathfrak{T}) = \{Z_1^{n_1} \cdots Z_g^{n_g} \mid (n_1, \dots, n_g) \in \mathbb{Z}^g\}.$$

In fact, a stronger statement is true: any holomorphic, nowhere vanishing function on \mathfrak{T} is a constant multiple of an algebraic character (see [13, §6.3]).

4.2. Consider the group homomorphism

$$\begin{aligned} \text{trop} : \mathfrak{T}(\mathbb{C}_K) &\rightarrow \text{Hom}(\mathcal{X}(\mathfrak{T}), \mathbb{R}) \approx \mathbb{R}^g \\ x &\mapsto (\chi \mapsto -\log |\chi(x)|). \end{aligned}$$

A (split) *lattice* Λ in \mathfrak{T} is a free rank- g subgroup of $\mathfrak{T}(K)$ such that $\text{trop} : \Lambda \rightarrow \mathbb{R}^g$ is injective and its image is a lattice in the classical sense. Such Λ is discrete in \mathfrak{T} , i.e., the intersection of Λ with any affinoid subset of \mathfrak{T} is finite. Hence we can form the quotient \mathfrak{T}/Λ in the usual way by gluing the Λ -translates of a small enough affinoid. The Riemann form condition in this setting is the following:

4.3. Theorem. *\mathfrak{T}/Λ is isomorphic to the rigid-analytification of an abelian variety over K if and only if there is a homomorphism*

$$H : \Lambda \rightarrow \mathcal{X}(\mathfrak{T})$$

such that $H(\lambda)(\mu) = H(\mu)(\lambda)$ for all $\lambda, \mu \in \Lambda$, and the symmetric bilinear form

$$\begin{aligned} \langle \cdot, \cdot \rangle_H : \Lambda \times \Lambda &\rightarrow \mathbb{Z} \\ \lambda, \mu &\mapsto \text{ord}_K H(\lambda)(\mu) \end{aligned}$$

is positive definite.

Proof. See [13, Ch. 6] or [4, §2]. □

4.4. Let A be an abelian variety of dimension g defined over K . We say that A is *uniformizable* if $A^{\text{an}} \cong \mathfrak{T}/\Lambda$ for some lattice Λ .

4.5. Theorem. *An abelian variety over K is uniformizable if and only if it has split toric reduction.*

Proof. See [4, §1]. □

4.6. If A has split toric reduction, then A^\vee also has split toric reduction; cf. (2.7). Let \mathfrak{T}/Λ be the uniformization of A . Denote

$$\mathfrak{T}^\vee = \text{Hom}(\Lambda, \mathbb{G}_{m,K}^{\text{an}}) \quad \text{and} \quad \Lambda^\vee = \text{Hom}(\mathfrak{T}, \mathbb{G}_{m,K}^{\text{an}}).$$

Note that Λ^\vee is the group of characters $\mathcal{X}(\mathfrak{T})$. We have a natural bilinear pairing $\Lambda^\vee \times \mathfrak{T}(K) \rightarrow K^\times$ given by evaluation of characters on the points of \mathfrak{T} . For a fixed $\lambda' \in \Lambda^\vee$, this pairing induces by restriction a homomorphism $\Lambda \rightarrow K^\times$, $\lambda \mapsto \lambda'(\lambda)$, and hence a K -valued point in \mathfrak{T}^\vee . If we vary $\lambda' \in \Lambda^\vee$, we obtain a canonical homomorphism $\Lambda^\vee \rightarrow \mathfrak{T}^\vee$, which is easy to see is the dual of $\Lambda \rightarrow \mathfrak{T}$. Hence Λ^\vee is naturally a lattice in \mathfrak{T}^\vee , and we can form the quotient $\mathfrak{T}^\vee/\Lambda^\vee$ as a proper rigid-analytic group. As one might expect, $\mathfrak{T}^\vee/\Lambda^\vee$ is canonically isomorphic to $(A^\vee)^{\text{an}}$; see [4, Thm. 2.1]. Let $H : \Lambda \rightarrow \Lambda^\vee$ be a Riemann form for A . Applying $\text{Hom}(\cdot, \mathbb{G}_{m,K}^{\text{an}})$ to H , we get a surjective homomorphism $H_{\mathfrak{T}} : \mathfrak{T} \rightarrow \mathfrak{T}^\vee$. From the definitions it is easy to see that the restriction of $H_{\mathfrak{T}}$ to $\Lambda \subset \mathfrak{T}$ is H . Hence we get a homomorphism $H_{A^{\text{an}}} : A^{\text{an}} \rightarrow (A^\vee)^{\text{an}}$. By GAGA, $H_{A^{\text{an}}}$ canonically corresponds to a homomorphism $H_A : A \rightarrow A^\vee$. Since H is injective with finite cokernel, H_A is an isogeny. In fact, one can show that H_A is a polarization and every polarization arises in this manner; cf. [4, §2].

4.7. More symmetrically, let Λ and Λ^\vee be two finitely generated free abelian groups of the same rank and let $[\cdot, \cdot] : \Lambda \times \Lambda^\vee \rightarrow K^\times$ be a bilinear pairing such that the pairing

$$\langle \cdot, \cdot \rangle = \text{ord}_K \circ [\cdot, \cdot] : \Lambda \times \Lambda^\vee \rightarrow \mathbb{Z}$$

becomes perfect after extending scalars from \mathbb{Z} to \mathbb{R} . Let $\mathfrak{T} = \text{Hom}(\Lambda^\vee, \mathbb{G}_{m,K}^{\text{an}})$ and $\mathfrak{T}^\vee = \text{Hom}(\Lambda, \mathbb{G}_{m,K}^{\text{an}})$. Then $[\cdot, \cdot]$ defines injective homomorphisms $\Lambda \hookrightarrow \mathfrak{T}(K)$ and $\Lambda^\vee \hookrightarrow \mathfrak{T}^\vee(K)$, the images of which are lattices. With these notations, a Riemann form is a homomorphism $H : \Lambda \rightarrow \Lambda^\vee$ such that $[\cdot, \cdot]_H = [\cdot, H(\cdot)]$ is symmetric and $\langle \cdot, \cdot \rangle_H = \langle \cdot, H(\cdot) \rangle$ is positive-definite. If such a form exists, then \mathfrak{T}/Λ and $\mathfrak{T}^\vee/\Lambda^\vee$ are dual abelian varieties.

4.8. Let $A_1^{\text{an}} = \mathfrak{T}_1/\Lambda_1$ and $A_2^{\text{an}} = \mathfrak{T}_2/\Lambda_2$ be uniformizable abelian varieties. Let $\text{Hom}(\mathfrak{T}_1, \Lambda_1; \mathfrak{T}_2, \Lambda_2)$ denote the group of homomorphisms $\varphi : \mathfrak{T}_1 \rightarrow \mathfrak{T}_2$ of analytic tori such that $\varphi(\Lambda_1) \subset \Lambda_2$. By a result of Gerritzen [16], the natural map

$$\text{Hom}(\mathfrak{T}_1, \Lambda_1; \mathfrak{T}_2, \Lambda_2) \rightarrow \text{Hom}(A_1, A_2)$$

is a bijection (see also [17, §7]).

Following the notations in (4.7), for $i = 1, 2$ let $\Lambda_i^\vee = \mathcal{X}(\mathfrak{T}_i)$, let \mathfrak{T}_i^\vee be the torus with character lattice Λ_i , let $[\cdot, \cdot]_i : \Lambda_i \times \Lambda_i^\vee \rightarrow K^\times$ denote the pairing induced by the inclusion $\Lambda_i \hookrightarrow \mathfrak{T}_i(K)$, and let $\langle \cdot, \cdot \rangle_i = \text{ord}_K \circ [\cdot, \cdot]_i$. Let $\varphi \in \text{Hom}(\mathfrak{T}_1, \Lambda_1; \mathfrak{T}_2, \Lambda_2)$. Then φ is determined by the induced homomorphism $\varphi^\vee : \Lambda_2^\vee \rightarrow \Lambda_1^\vee$ of character groups, and since $\varphi(\Lambda_1) \subset \Lambda_2$, we have

$$(4.8.1) \quad [\varphi(\lambda_1), \lambda_2^\vee]_2 = [\lambda_1, \varphi^\vee(\lambda_2^\vee)]_1$$

for all $\lambda_1 \in \Lambda_1$ and $\lambda_2^\vee \in \Lambda_2^\vee$. We can therefore define $\text{Hom}(\mathfrak{T}_1, \Lambda_1; \mathfrak{T}_2, \Lambda_2)$ more symmetrically as the group of pairs (φ, φ^\vee) of homomorphisms $\varphi : \Lambda_1 \rightarrow \Lambda_2$ and $\varphi^\vee : \Lambda_2^\vee \rightarrow \Lambda_1^\vee$ satisfying (4.8.1). Since $\langle \cdot, \cdot \rangle_i$ is nondegenerate for $i = 1, 2$, it is clear that φ and φ^\vee determine each other. If $(\varphi, \varphi^\vee) \in \text{Hom}(\mathfrak{T}_1, \Lambda_1; \mathfrak{T}_2, \Lambda_2)$ corresponds to the homomorphism $f : A_1 \rightarrow A_2$ then $(\varphi^\vee, \varphi) \in \text{Hom}(\mathfrak{T}_2^\vee, \Lambda_2^\vee; \mathfrak{T}_1^\vee, \Lambda_1^\vee)$ corresponds to the dual homomorphism $f^\vee : A_2^\vee \rightarrow A_1^\vee$.

Now let $H_i : \Lambda_i \xrightarrow{\sim} \Lambda_i^\vee$ be Riemann forms determining principal polarizations $A_i \xrightarrow{\sim} A_i^\vee$ for $i = 1, 2$. Using H_i to identify Λ_i with Λ_i^\vee , we can describe an element of $\text{Hom}(\mathfrak{T}_1, \Lambda_1; \mathfrak{T}_2, \Lambda_2)$ as a pair (φ, φ^\vee) , where $\varphi : \Lambda_1 \rightarrow \Lambda_2$ and $\varphi^\vee : \Lambda_2 \rightarrow \Lambda_1$ are homomorphisms satisfying

$$(4.8.2) \quad [\varphi(\lambda_1), \lambda_2]_{H_2} = [\lambda_1, \varphi^\vee(\lambda_2)]_{H_1}$$

for all $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$. As above, if (φ, φ^\vee) corresponds to the homomorphism $f : A_1 \rightarrow A_2$ then (φ^\vee, φ) corresponds to the dual homomorphism $f^\vee : A_2 \cong A_2^\vee \rightarrow A_1^\vee \cong A_1$.

4.9. Proposition. *Assume $A^{\text{an}} \cong \mathfrak{T}/\Lambda$ is a principally polarizable abelian variety. Fix a principal polarization $H : \Lambda \xrightarrow{\sim} \mathcal{X}(\mathfrak{T})$. An endomorphism $T \in \text{End}(A)$ induces an endomorphism of Λ , which we denote by the same letter. Let $T^\dagger \in \text{End}(A)$ be the image of T under the Rosati involution with respect to the principal polarization H . Then for any $\lambda, \mu \in \Lambda$,*

$$H(T\lambda)(\mu) = H(\lambda)(T^\dagger\mu).$$

Proof. Let $\Lambda^\vee = \mathcal{X}(\mathfrak{T})$ and let $[\cdot, \cdot] : \Lambda \times \Lambda^\vee \rightarrow K^\times$ be the pairing induced by the inclusion $\Lambda \hookrightarrow \mathfrak{T}(K)$, as in (4.7). By (4.8), we can describe T as a pair of

endomorphisms $\varphi, \varphi^\vee : \Lambda \rightarrow \Lambda$ satisfying

$$H(\varphi(\lambda))(\lambda') = [\varphi(\lambda), \lambda']_H = [\lambda, \varphi^\vee(\lambda')]_H = H(\lambda)(\varphi^\vee(\lambda'))$$

for all $\lambda, \lambda' \in \Lambda$. The endomorphism T^\dagger then corresponds to the pair (φ^\vee, φ) . Under these identifications, the endomorphism of Λ induced by T (resp. T^\dagger) is exactly φ (resp. φ^\vee). \square

4.10. Let $J := \text{Pic}_{X/K}^0$ be the Jacobian variety of a curve X over K . Assume J has split toric reduction. Let H be the canonical principal polarization on J . The uniformization of J is given by

$$0 \rightarrow \Lambda \xrightarrow{H} \text{Hom}(\Lambda, \mathbb{C}_K^\times) \rightarrow J(\mathbb{C}_K) \rightarrow 0.$$

Let E be an elliptic curve which is an optimal quotient $\pi : J \rightarrow E$. Using the canonical principal polarizations on E and J , we can consider E as an abelian subvariety of J via the dual morphism $\pi^\vee : E \hookrightarrow J$; cf. (2.11). Sometimes to emphasize that we consider E as the image of π (resp. the domain of π^\vee) we will write E_* (resp. E^*).

To simplify the notation, we will denote the pairing $\langle \cdot, \cdot \rangle_H$ of Theorem 4.3 for the canonical principal polarization on J by $\langle \cdot, \cdot \rangle$. Likewise we denote the pairing $[\cdot, \cdot]_H : \Lambda \times \Lambda \rightarrow K^\times$ of (4.7) by $[\cdot, \cdot]$.

4.11. Since E is a subvariety of J , it has split toric reduction; cf. (2.7). Therefore E is uniformizable:

$$(4.11.1) \quad 0 \rightarrow \Gamma \rightarrow \mathbb{C}_K^\times \rightarrow E(\mathbb{C}_K) \rightarrow 0,$$

where Γ , as a subgroup of \mathbb{C}_K^\times , is $q_E^\mathbb{Z}$ for some $q_E \in \mathbb{C}_K^\times$ with $\text{ord}_K(q_E) > 0$. More precisely, since E carries a canonical principal polarization, it is uniformized by the torus $\text{Hom}(\Gamma, \mathbb{C}_K^\times)$; fixing a generator ρ of Γ , we identify $\text{Hom}(\Gamma, \mathbb{C}_K^\times)$ with \mathbb{C}_K^\times via the isomorphism $f \mapsto f(\rho)$. By (4.8), the closed immersion $\pi^\vee : E \rightarrow J$ induces a homomorphism $\pi^\vee : \Gamma \rightarrow \Lambda$ and a homomorphism of tori $\mathbb{C}_K^\times \rightarrow \text{Hom}(\Lambda, \mathbb{C}_K^\times)$ making following diagram commute:

$$(4.11.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Gamma & \longrightarrow & \mathbb{C}_K^\times & \longrightarrow & E(\mathbb{C}_K) \longrightarrow 0 \\ & & \pi^\vee \downarrow & & \downarrow & & \downarrow \pi^\vee \\ 0 & \longrightarrow & \Lambda & \longrightarrow & \text{Hom}(\Lambda, \mathbb{C}_K^\times) & \longrightarrow & J(\mathbb{C}_K) \longrightarrow 0 \end{array}$$

It is easy to see that the vertical arrows in (4.11.2) are injective. In principle, $\pi^\vee(\Gamma)$ need not be saturated in Λ , i.e., the abelian group $\Lambda/\pi^\vee(\Gamma)$ might have non-trivial torsion. Let Γ' be the saturation of $\pi^\vee(\Gamma)$ in Λ . We can write

$$\pi^\vee(\rho) = c \cdot \lambda_E,$$

where c is a uniquely determined positive integer, λ_E is a generator of Γ' , and ρ is our fixed generator of Γ .

4.12. Let $\pi : \Lambda \rightarrow \Gamma$ be the homomorphism of character groups associated to the middle vertical arrow of (4.11.2). The homomorphism $\pi^\vee : \Gamma \rightarrow \Lambda$ induces the homomorphism of tori $\text{ev}_\rho : \text{Hom}(\Lambda, \mathbb{C}_K^\times) \rightarrow \text{Hom}(\Gamma, \mathbb{C}_K^\times) = \mathbb{C}_K^\times$ given by

$\text{ev}_\rho(f) = f(\pi^\vee(\rho))$. By the discussion in (4.8), the following diagram commutes:

$$(4.12.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & \text{Hom}(\Lambda, \mathbb{C}_K^\times) & \longrightarrow & J(\mathbb{C}_K) \longrightarrow 0 \\ & & \downarrow \pi & & \downarrow \text{ev}_\rho & & \downarrow \pi \\ 0 & \longrightarrow & \Gamma & \longrightarrow & \mathbb{C}_K^\times & \longrightarrow & E(\mathbb{C}_K) \longrightarrow 0 \end{array}$$

It is easy to see that the vertical arrows in (4.12.1) are surjective.

4.13. Let $c^{-1}\Gamma = \{x \in \mathbb{C}_K^\times \mid x^c \in \Gamma\}$. Since $\Gamma = q_E^\mathbb{Z}$ we have $c^{-1}\Gamma = \mu_c \times w^\mathbb{Z}$, where $\mu_c \subset \mathbb{C}_K^\times$ is the group of c th roots of unity and w is any c th root of q_E . In particular,

$$(4.13.1) \quad \text{ord}_K(q_E) = c \cdot \text{ord}_K(w).$$

Define $\text{ev}_E : \text{Hom}(\Lambda, \mathbb{C}_K^\times) \rightarrow \mathbb{C}_K^\times$ by $\text{ev}_E(f) = f(\lambda_E)$. Then $\text{ev}_E^c = \text{ev}_\rho$, so we have a commutative diagram

$$(4.13.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & \text{Hom}(\Lambda, \mathbb{C}_K^\times) & \longrightarrow & J(\mathbb{C}_K) \longrightarrow 0 \\ & & \downarrow & & \downarrow \text{ev}_E & & \downarrow \pi \\ 0 & \longrightarrow & c^{-1}\Gamma & \longrightarrow & \mathbb{C}_K^\times & \longrightarrow & E(\mathbb{C}_K) \longrightarrow 0 \end{array}$$

where the map $\mathbb{C}_K^\times \rightarrow E(\mathbb{C}_K)$ in (4.13.2) is the c th power of the one in (4.12.1). We claim that the vertical arrows in (4.13.2) are again surjective. Since ev_E is surjective, by the snake lemma it suffices to prove that $\ker(\text{ev}_E) \rightarrow \ker(\pi)$ is surjective. Let $x \in \ker(\pi)$. Since $\ker(\pi)$ is an abelian subvariety of J , it is divisible; choose $y \in \ker(\pi)$ such that $cy = x$. Since $\ker(\text{ev}_\rho)$ surjects onto $\ker(\pi)$ there exists $z \in \ker(\rho)$ such that $z^c \mapsto x$ and $\text{ev}_E(z^c) = \text{ev}_\rho(z) = 1$, which proves the claim. This implies

$$c^{-1}\Gamma = \{[\lambda, \lambda_E] \mid \lambda \in \Lambda\} \subset K^\times.$$

In particular, c divides the order of the group of roots of unity in K .

4.14. The endomorphism

$$e_0 = \pi^\vee \circ \pi : J \rightarrow J$$

corresponds to an idempotent $e \in \text{End}^0(J) := \text{End}(J) \otimes_{\mathbb{Z}} \mathbb{Q}$. Up to isogeny, we can decompose

$$J \sim_K A_1 \times A_2 \times \cdots \times A_s,$$

where A_i 's are K -simple abelian varieties. This decomposition produces idempotents

$$e_1, \dots, e_s \in \text{End}^0(J)$$

which are mutually orthogonal: $e_i e_j = 0$ if $i \neq j$. The idempotent e is one of those. The \mathbb{Q} -bilinear form $B(x, y) = \text{Tr}(xy^\dagger)$ on $\text{End}^0(A)$ is symmetric and positive definite (here the Rosati involution is with respect to the canonical principal polarization H). This implies that the Rosati involution must fix each idempotent e_i . Therefore $e^\dagger = e$, and also $e_0^\dagger = e_0$. This observation will simplify some calculations and is useful in the following paragraph.

We denote by n the denominator of e in $\text{End}(J)$, i.e., the least natural number such that $ne \in \text{End}(J)$. Note that (4.8) implies that $\text{End}(J)$ is naturally a subring of $\text{End}(\Lambda)$ when we regard Λ as the lattice uniformizing J , and $\text{End}(J)$ is a subring of $\text{End}(\Lambda)^{\text{opp}}$ when we regard Λ as the character group of the torus uniformizing

J . By Proposition 4.9 and the above discussion, the image of e_0 in $\text{End}(\Lambda)$ is the same under either identification. We define the denominator r of e in $\text{End}(\Lambda)$ as the least natural number such that $re \in \text{End}(\Lambda)$. Obviously, r divides n .

4.15. Lemma. *The morphism $\pi \circ \pi^\vee : E^* \rightarrow E_*$ is the multiplication-by- n map on E .*

Proof. See the proof of Theorem 3 in [35]. \square

4.16. Recall that the closed immersion $E \hookrightarrow J$ gives rise to the inclusion $\pi^\vee : \Gamma \hookrightarrow \Lambda$ sending $\rho \mapsto c\lambda_E$, and that the projection $\pi : J \rightarrow E$ induces a surjective homomorphism $\pi : \Lambda \twoheadrightarrow \Gamma$. The endomorphism $\pi \circ \pi^\vee : E^* \rightarrow E_*$ corresponds to the endomorphism $\pi \circ \pi^\vee : \Gamma \hookrightarrow \Lambda \twoheadrightarrow \Gamma$, so by Lemma 4.15, $\pi(c\lambda_E) = \pi \circ \pi^\vee(\rho) = n\rho$, and therefore

$$(4.16.1) \quad \pi(\lambda_E) = \frac{n}{c}\rho.$$

The idempotent e_0 corresponds to the composition $\pi^\vee \circ \pi : \Lambda \twoheadrightarrow \Gamma \hookrightarrow \Lambda$. We have $\pi^\vee \circ \pi(\lambda_E) = \pi^\vee(\frac{n}{c}\rho) = n\lambda_E$, so $e_0 = ne$ because $e(\lambda_E) = \lambda_E$. Since $\frac{1}{c}\pi^\vee(\Gamma) \subset \Lambda$ but $\frac{1}{c'}\pi^\vee(\Gamma) \not\subset \Lambda$ for $c' > c$, we have $\frac{1}{c}e_0 \in \text{End}(\Lambda)$ but $\frac{1}{c'}e_0 \notin \text{End}(\Lambda)$ for $c' > c$. Thus $re = \frac{1}{c}e_0 = \frac{n}{c}e$, i.e.

$$(4.16.2) \quad c = \frac{n}{r}$$

4.17. The pairing $\langle \cdot, \cdot \rangle$ coincides with the (H -polarized version of) Grothendieck's monodromy pairing: see [17, (14.2.5)] and [7]. By [17, (11.5)] the cokernel of the map $\Lambda \rightarrow \text{Hom}(\Lambda, \mathbb{Z})$ induced by the monodromy pairing $\langle \cdot, \cdot \rangle$ is naturally isomorphic to the component group Φ_J . The analogous statement holds for E , and we have a commutative diagram

$$(4.17.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Lambda & \xrightarrow{\langle \cdot, \cdot \rangle} & \text{Hom}(\Lambda, \mathbb{Z}) & \longrightarrow & \Phi_J \longrightarrow 0 \\ & & \downarrow \pi & & \downarrow \text{ev}_\rho & & \downarrow \pi_* \\ 0 & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z} & \longrightarrow & \Phi_E \longrightarrow 0 \end{array}$$

where $\text{ev}_\rho(f) = f(\pi^\vee(\rho))$ as in (4.12.1). Since $\pi^\vee(\rho) = c\lambda_E$ and $\mathbb{Z}\lambda_E$ is a direct summand of Λ , the cokernel of ev_ρ is isomorphic to $\mathbb{Z}/c\mathbb{Z}$. As $\pi : \Lambda \rightarrow \Gamma$ is surjective, this implies that

$$(4.17.2) \quad \text{coker}(\pi_* : \Phi_J \rightarrow \Phi_E) \cong \mathbb{Z}/c\mathbb{Z}.$$

This is a generalization of Formula 1 in [28].

4.18. *Corollary.* $\#\text{coker}(\pi_*)$ divides the order of the group of roots of unity in K^\times .

4.19. The map $\Gamma \rightarrow \mathbb{Z}$ is the composition of $\Gamma \rightarrow K^\times$ with $\text{ord}_K : K^\times \rightarrow \mathbb{Z}$; hence ρ maps to $\text{ord}_K(q_E)$. (This recovers the well-known fact that $\#\Phi_E = \text{ord}_K(q_E)$.) We have $\rho = \frac{c}{n}\pi(\lambda_E)$ by (4.16.1), so since the left square commutes,

$$c\langle \lambda_E, \lambda_E \rangle = \langle \lambda_E, \pi^\vee(\rho) \rangle = \frac{n}{c}\text{ord}_K(q_E),$$

and therefore,

$$(4.19.1) \quad c^2 \langle \lambda_E, \lambda_E \rangle = n \text{ord}_K(q_E).$$

This is essentially Formula 3 in [28].

4.20. Let

$$m := \min\{\langle \lambda, \lambda_E \rangle > 0 \mid \lambda \in \Lambda\};$$

$$\lambda_E^\perp := \{\lambda \in \Lambda \mid \langle \lambda, \lambda_E \rangle = 0\}.$$

The image of $\text{ev}_\rho \circ \langle \cdot, \cdot \rangle$ in (4.17.1) is exactly $\min\{\langle \lambda, c\lambda_E \rangle > 0 \mid \lambda \in \Lambda\} = cm$; as $\pi : \Lambda \rightarrow \Gamma$ is surjective and the image of Γ in \mathbb{Z} is generated by $\text{ord}_K(q_E)$, this implies

$$(4.20.1) \quad c \cdot m = \text{ord}_K(q_E).$$

4.21. **Theorem.** *The following are equivalent:*

- (1) *The functorially induced map on component groups $\Phi_J \rightarrow \Phi_E$ is surjective.*
- (2) *e_0 is primitive in $\text{End}(\Lambda)$.*
- (3) *$c = 1$.*
- (4) *$n = r$.*
- (5) *$\langle \lambda_E, \lambda_E \rangle = n \text{ord}_K(q_E)$.*
- (6) *$m = \text{ord}_K(q_E)$.*
- (7) *$n = [\Lambda : \lambda_E^\perp \oplus \mathbb{Z}\lambda_E]$.*

Proof. We have (1) \iff (3) by (4.17.2), (3) \iff (4) by (4.16.2), and (4) \iff (2) since $e_0 = ne$. Conditions (5) and (6) are equivalent to (3) by (4.19.1) and (4.20.1), respectively. It is easy to see that $r = [\Lambda : \lambda_E^\perp \oplus \mathbb{Z}\lambda_E]$, hence (4) \iff (7). \square

4.22. *Remark.* Assuming X has a K -rational point, one can use the Abel-Jacobi map to realize X as a closed subvariety of J , $\theta : X \hookrightarrow J$. The composition $X \xrightarrow{\theta} J \xrightarrow{\pi} E$ is a non-constant morphism since $\theta(X)$ generates J . It is not hard to show that $n = \deg(\pi \circ \theta)$; cf. [35]. The index $[\Lambda : \lambda_E^\perp \oplus \mathbb{Z}\lambda_E]$ is the “congruence number” of λ_E with respect to the monodromy pairing. Hence Theorem 4.21 implies that the degree of $X \rightarrow E$ is divisible by the congruence number and the ratio is c . It is interesting to compare this fact with the well-known relation between the degree of modular parametrization of an elliptic curve and the congruence number of the corresponding newform in the space of weight-2 cusp forms with integer Fourier coefficients; cf. [1].

We use Theorem 4.21 to give two conditions under which $\Phi_J \rightarrow \Phi_E$ is surjective. Suppose there is a commutative subring $\mathbb{T} \subset \text{End}(J)$ with the same identity element and such that $e \in \mathbb{T} \otimes \mathbb{Q}$. The reader might guess from our notation that the situation we have in mind is when J is the Jacobian of some modular curve and \mathbb{T} is a Hecke algebra.

4.23. **Lemma.** *Suppose there is a bilinear \mathbb{T} -equivariant pairing*

$$(\cdot, \cdot) : \mathbb{T} \times \Lambda \rightarrow \mathbb{Z}$$

which is perfect if we consider \mathbb{T} and Λ as free \mathbb{Z} -modules. Then the equivalent conditions of Theorem 4.21 are satisfied.

Proof. Let s be the denominator of e in \mathbb{T} . Because $se \in \mathbb{T}$ is primitive, we can take it as part of a \mathbb{Z} -basis of \mathbb{T} . On the other hand, for any $\lambda \in \Lambda$

$$(se, \lambda) = (1, (se)\lambda) = \frac{s}{r}(1, (re)\lambda) \in \frac{s}{r}\mathbb{Z}.$$

Hence s/r divides the determinant of (\cdot, \cdot) with respect to some \mathbb{Z} -bases of \mathbb{T} and Λ . The perfectness of the pairing is equivalent to this determinant being ± 1 , so

$s = r$. But r divides n and n divides s , since $\mathbb{T} \subset \text{End}(J) \subset \text{End}(\Lambda)$. Therefore, $r = n = s$. \square

4.24. As is easy to check, the assumption $e \in \mathbb{T} \otimes \mathbb{Q}$ implies that Γ' is \mathbb{T} -invariant, that is, for any $T \in \mathbb{T}$ we have $T\lambda_E = a(T) \cdot \lambda_E$ for some $a(T) \in \mathbb{Z}$. It is clear that the map $T \mapsto a(T)$ gives a homomorphism $\mathbb{T} \rightarrow \mathbb{Z}$. Denote the kernel of this homomorphism by I_E . Define

$$I_E\Lambda = \{T\lambda \mid T \in I_E, \lambda \in \Lambda\} = \{T\lambda - a(T)\lambda \mid T \in \mathbb{T}, \lambda \in \Lambda\}.$$

Assume $a(T^\dagger) = a(T)$ for all $T \in \mathbb{T}$. Since

$$\langle T\lambda - a(T)\lambda, \lambda_E \rangle = \langle \lambda, T^\dagger \lambda_E \rangle - a(T)\langle \lambda, \lambda_E \rangle = 0,$$

we have an inclusion $I_E\Lambda \subset \lambda_E^\perp$.

4.25. **Lemma.** *If $I_E\Lambda = \lambda_E^\perp$ then the equivalent conditions of Theorem 4.21 are satisfied.*

Proof. Let $\lambda \in \Lambda$ be such that $H(\lambda)(\lambda_E)$ generates μ_c in $c^{-1}\Gamma \cong \mu_c \times w^\mathbb{Z}$. Then

$$\text{ord}_K H(\lambda)(\lambda_E) = \langle \lambda, \lambda_E \rangle = 0,$$

which implies $\lambda \in \lambda_E^\perp$. By assumption, there exist $T \in \mathbb{T}$ and $\lambda' \in \Lambda$ such that $\lambda = T\lambda' - a(T)\lambda'$. Combining this with Proposition 4.9, we get

$$\begin{aligned} H(\lambda)(\lambda_E) &= H(T\lambda' - a(T)\lambda')(\lambda_E) = H(T\lambda')(\lambda_E)/H(\lambda')(\lambda_E)^{a(T)} \\ &= H(\lambda')(T^\dagger \lambda_E)/H(\lambda')(\lambda_E)^{a(T)} = H(\lambda')(\lambda_E)^{a(T)}/H(\lambda')(\lambda_E)^{a(T)} = 1. \end{aligned}$$

Hence $c = 1$. \square

Now we discuss some examples where conditions similar to those in Lemma 4.23 and Lemma 4.25 are satisfied.

4.26. Let $A = \mathbb{F}[T]$ be the ring of polynomials with coefficients in a finite field \mathbb{F} . Let $K = \mathbb{F}((1/T))$ be completion of the fraction field of A at $1/T$. Let $\mathfrak{n} \triangleleft A$ be an ideal and

$$\Gamma_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(A) \mid c \in \mathfrak{n} \right\}.$$

The group $\Gamma_0(\mathfrak{n})$ acts discontinuously on the Drinfeld half plane $\Omega := \mathbb{C}_K - K$, and the quotient $\Gamma_0(\mathfrak{n}) \backslash \Omega$ is the analytification of the Drinfeld modular curve $Y_0(\mathfrak{n})$, which is a smooth affine algebraic curve defined over K ; see [11]. The \mathbb{C}_K -valued points of $Y_0(\mathfrak{n})$ are in bijection with rank-2 Drinfeld A -modules over \mathbb{C}_K with certain level structures. The existence of analytic uniformization of $Y_0(\mathfrak{n})$ implies that the Jacobian $J_0(\mathfrak{n})$ of the projective curve containing $Y_0(\mathfrak{n})$ as a dense subset has split toric reduction. The Hecke operators defined in terms of correspondences on $Y_0(\mathfrak{n})$ generate a commutative algebra $\mathbb{T} \subset \text{End}(J_0(\mathfrak{n}))$. There is a canonical \mathbb{T} -equivariant isomorphism between the uniformizing lattice Λ of $J_0(\mathfrak{n})$ and the group of \mathbb{Z} -valued $\Gamma_0(\mathfrak{n})$ -invariant harmonic cochains $H_1(\mathcal{T}, \mathbb{Z})^{\Gamma_0(\mathfrak{n})}$ on the Bruhat-Tits tree \mathcal{T} of $\text{PGL}_2(K)$; see [14, Thm. 1.9] and [15, Lem. 9.3.2]. The proof that Λ and $H_1(\mathcal{T}, \mathbb{Z})^{\Gamma_0(\mathfrak{n})}$ are isomorphic \mathbb{T} -modules relies on a rather subtle analysis of the quotient tree $\Gamma_0(\mathfrak{n}) \backslash \mathcal{T}$. As is explained in [15], $H_1(\mathcal{T}, \mathbb{C})^{\Gamma_0(\mathfrak{n})}$ may be interpreted as a space of automorphic forms in the sense of Jacquet and Langlands. From this perspective, $H_1(\mathcal{T}, \mathbb{Z})^{\Gamma_0(\mathfrak{n})}$ is an integral structure on the space of automorphic forms, stable under the action of Hecke operators. Weil developed a theory of

Fourier expansions of automorphic forms over function fields. In [14], using these Fourier expansions, Gekeler defines a pairing

$$\mathbb{T} \times H_1(\mathcal{T}, \mathbb{Z})^{\Gamma_0(\mathfrak{n})} \rightarrow \mathbb{Z}$$

and proves that it is perfect after tensoring with $\mathbb{Z}[p^{-1}]$, where p is the characteristic of \mathbb{F} . (This pairing is the function field analogue of the well-known perfect pairing between the Hecke algebra and the group of weight-2 cusp forms on $\Gamma_0(N)$ with integral q -expansions; see Theorem 2.2 in [27].) Now let E be an optimal quotient of $J_0(\mathfrak{n})$. It is not hard to show that the idempotent e always lies in $\mathbb{T} \otimes \mathbb{Q}$, and since $\Lambda \cong H_1(\mathcal{T}, \mathbb{Z})^{\Gamma_0(\mathfrak{n})}$ as \mathbb{T} -modules, the argument in the proof of Lemma 4.23 shows that c is a p -power. On the other hand, c divides $\#\mathbb{F} - 1$, so c is coprime to p . Overall, we conclude that the induced map on component groups $\Phi_{J_0(\mathfrak{n})} \rightarrow \Phi_E$ is always surjective.

4.27. Consider the modular curve $X_0(p)$ classifying elliptic curves with cyclic subgroups of order p , where p is prime. According to Deligne and Rapoport [10] this curve has degenerate reduction over $K = \mathbb{Q}_p$. Let $J_0(p)$ be the Jacobian of $X_0(p)$, and $\pi : J_0(p) \rightarrow E$ be an optimal quotient. It is proven in [25, Cor. 3], and also in [12], that the induced map on component groups $\pi_* : \Phi_{J_0(p)} \rightarrow \Phi_E$ of the Néron models over \mathbb{Z}_p is surjective. Both proofs rely on Ribet's level-lowering theorem [29], and the deepest results in [24]. Let \mathbb{T} be the Hecke algebra acting on the space $S_2(\Gamma_0(p))$ of weight-2 cusp forms on $\Gamma_0(p)$. We may identify $S_2(\Gamma_0(p))$ with the space of invariant differentials on $J_0(p)$, so \mathbb{T} can be realized as a subalgebra of $\text{End}(J_0(p))$. It is easy to see that λ_E^\perp is stable under the action of \mathbb{T} , hence $\lambda_E^\perp/I_E\Lambda$ is a \mathbb{T} -module. In [12], Emerton shows that the surjectivity of π_* follows from the triviality of a finite \mathbb{T} -module similar to $\lambda_E^\perp/I_E\Lambda$. He then proves that this latter module is supported at the maximal ideals \mathfrak{m} of \mathbb{T} which are Eisenstein but the completion $\mathbb{T}_{\mathfrak{m}}$ is not Gorenstein. Since by [24] there are no such ideals, the surjectivity of π_* follows.

4.28. Let $D > 1$ be a square-free integer divisible by an even number of primes, and $M \geq 1$ be an integer coprime to D . Let $\Gamma_0^D(M)$ be the group of norm-1 units in an Eichler order of level M in the indefinite quaternion algebra B over \mathbb{Q} of discriminant D . Since B is indefinite, by fixing an isomorphism $B \otimes \mathbb{R} \cong \mathbb{M}_2(\mathbb{R})$, we can regard $\Gamma_0^D(M)$ as a discrete subgroup of $\text{SL}_2(\mathbb{R})$. Let $X_0^D(M) = \Gamma_0^D(M) \backslash \mathcal{H}$ be the associated Shimura curve, where $\mathcal{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$. This is a smooth projective curve, which has a canonical model over \mathbb{Q} . It is a moduli space of abelian surfaces equipped with an action of B and $\Gamma_0(M)$ -level structure. Let $J_0^D(M)$ be the Jacobian of $X_0^D(M)$. By the work of Cherednik and Drinfeld it is known that for any $p \nmid D$ the curve $X_0^D(M)$ over \mathbb{Q}_p has degenerate reduction; cf. [6]. Let \mathbb{T} be the Hecke algebra acting on the space $S_2(\Gamma_0^D(M))$ of weight-2 cusp forms on $\Gamma_0^D(M)$. As in (4.27), \mathbb{T} acts on $J_0^D(M)$. To arrive at the setting of this section, fix a prime $p \nmid D$ and consider an optimal quotient $\pi : J_0^D(M) \rightarrow E$ over $K = \mathbb{Q}_p$. In the proof of Proposition 4.4 in [3] the authors implicitly assume that the equality $I_E\Lambda = \lambda_E^\perp$ holds. It is not clear if this is true in general.

4.29. The elliptic curve E in (4.28) can be defined over \mathbb{Q} . Let ℓ be an arbitrary prime and let p be a prime dividing D . In a positive direction for the question of surjectivity of $\pi_* : \Phi_{J_0^D(M)} \rightarrow \Phi_E$ over $\overline{\mathbb{F}}_p$, Ribet and Takahashi proved that if M is square-free and $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -module $E[\ell]$ is irreducible, then ℓ does not divide

c ; see [30] and [33]. The proof relies on the comparison of the degrees of different modular parametrizations of E .

4.30. *Remark.* Theorem 4.21 suggests a computational approach to finding an example of an optimal quotient E of a Shimura curve $X_0^D(M)$ such that the homomorphism π_* of component groups is not surjective. The computer algebra package Magma has an implementation of Brandt modules, which allows one to do calculations with the lattice Λ uniformizing the analytification of $J_0^D(M)$. In particular, one can efficiently calculate the idempotent e . The surjectivity question then reduces to whether or not re , as an endomorphism of $\text{Hom}(\Lambda, K^\times)$, takes Λ to itself. This calculation can in theory be carried out using p -adic Θ -functions.

5. JACOBIANS ISOGENOUS TO A PRODUCT OF TWO ELLIPTIC CURVES: ANALYTIC APPROACH

In this section we give an analytic construction of (a slight generalization of) the example in Section 3, in order to give a better understanding of that example and also to illustrate the machinery of Section 4.

5.1. Let $\mathfrak{T} = (\mathbb{G}_{m,K}^2)^{\text{an}}$ be a two-dimensional split analytic torus over K . Fix $q_1, q_2 \in K^\times$ such that $\text{ord}_K(q_1), \text{ord}_K(q_2) > 0$ and $q_1^u \neq q_2^w$ for any non-zero $u, w \in \mathbb{Z}$. Let $c > 1$ be an integer and let $\zeta \in K^\times$ be a c th root of unity. Let $\Lambda \subset \mathfrak{T}(K) = (K^\times)^2$ be the free abelian group generated by (q_1, ζ) and (ζ, q_2) . We have $\text{trop}(q_1, \zeta) = (-\log|q_1|, 0)$ and $\text{trop}(\zeta, q_2) = (0, -\log|q_2|)$ which are linearly independent in \mathbb{R}^2 , so Λ is a lattice in \mathfrak{T} . Let J^{an} be the analytic quotient \mathfrak{T}/Λ .

5.2. We identify $(n_1, n_2) \in \mathbb{Z}^2$ with the character of \mathfrak{T} defined by $(Z_1, Z_2) \mapsto Z_1^{n_1} Z_2^{n_2}$. Define $H : \Lambda \xrightarrow{\sim} \mathbb{Z}^2$ by

$$H(q_1, \zeta) = (1, 0) \quad \text{and} \quad H(\zeta, q_2) = (0, 1).$$

We have

$$H(q_1, \zeta)(q_1, \zeta) = q_1 \quad H(q_1, \zeta)(\zeta, q_2) = \zeta = H(\zeta, q_2)(q_1, \zeta) \quad H(\zeta, q_2)(\zeta, q_2) = q_2,$$

so $H(\lambda)(\mu) = H(\mu)(\lambda)$ for all $\lambda, \mu \in \Lambda$. Moreover, the symmetric bilinear form $\langle \cdot, \cdot \rangle_H$ has the matrix form $\begin{bmatrix} \text{ord}_K(q_1) & 0 \\ 0 & \text{ord}_K(q_2) \end{bmatrix}$ with respect to the above choice of basis, so $\langle \cdot, \cdot \rangle_H$ is positive definite. Therefore by Theorem 4.3, J^{an} is the analytification of an abelian variety J , and the Riemann form H gives rise to a principal polarization of J by (4.6).

5.3. By an *elliptic subvariety* of J we will mean an abelian subvariety E of J of dimension one. By (2.7), any elliptic subvariety of J has split multiplicative reduction; moreover, if $0 \rightarrow \Gamma \rightarrow \mathbb{C}_K^\times \rightarrow E(\mathbb{C}_K) \rightarrow 0$ is the Tate uniformization of E then we have a homomorphism of short exact sequences

$$(5.3.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Gamma & \longrightarrow & \mathbb{C}_K^\times & \longrightarrow & E(\mathbb{C}_K) \longrightarrow 0 \\ & & \downarrow & & \downarrow \varphi & & \downarrow \\ 0 & \longrightarrow & \Lambda & \longrightarrow & (\mathbb{C}_K^\times)^2 & \longrightarrow & J(\mathbb{C}_K) \longrightarrow 0 \end{array}$$

with injective vertical arrows. In particular, $\varphi(\mathbb{C}_K^\times) \cap \Lambda = \varphi(\Gamma)$. Conversely, let $\mathbb{G}_{m,K}^{\text{an}} \cong \mathfrak{T}' \subset \mathfrak{T}$ be a subtorus of dimension one such that $\Gamma = \mathfrak{T}'(K) \cap \Lambda \cong \mathbb{Z}$ (equivalently, such that $\mathfrak{T}'(\mathbb{C}_K) \cap \Lambda \neq \{1\}$), and let $E^{\text{an}} = \mathfrak{T}'/\Gamma$. Then E^{an} is the

analytification of an elliptic curve E over K and the induced map $E \rightarrow J$ is a closed immersion, so E is an elliptic subvariety of J and the diagram (5.3.1) commutes.

5.4. Proposition. *Let J be as in (5.1). There are exactly two elliptic subvarieties of J , given by*

$$E_1(\mathbb{C}_K) = \mathbb{C}_K^\times \times \{1\} / (q_1^c, 1)^{\mathbb{Z}} \quad \text{and} \quad E_2(\mathbb{C}_K) = \{1\} \times \mathbb{C}_K^\times / (1, q_2^c)^{\mathbb{Z}}.$$

Proof. It is clear that E_1 and E_2 are elliptic subvarieties of J . Any dimension-one subtorus \mathfrak{T}' of \mathfrak{T} is of the form

$$\mathfrak{T}'(\mathbb{C}_K) = \{(z, w) \mid z^\alpha w^\beta = 1\}$$

for some coprime integers $\alpha, \beta \in \mathbb{Z}$. Let \mathfrak{T}' be such a subtorus, and suppose that $\mathfrak{T}'(K) \cap \Lambda \neq \{1\}$. Let $\lambda \in \Lambda \setminus \{1\}$ be an element of $\mathfrak{T}'(K) \cap \Lambda$. Then

$$\lambda = (q_1, \zeta)^\gamma (\zeta, q_2)^\delta = (q_1^\gamma \zeta^\delta, q_2^\delta \zeta^\gamma)$$

for some integers γ, δ , not both equal to zero, and we have

$$q_1^{\alpha\gamma} q_2^{\beta\delta} \zeta^{\alpha\delta + \beta\gamma} = 1.$$

Raising both sides to the c th power gives $q_1^{\alpha\gamma c} q_2^{\beta\delta c} = 1$, so we must have $\alpha\gamma = \beta\delta = 0$ by the way we chose q_1, q_2 . If $\alpha \neq 0$ and $\beta \neq 0$ then $\gamma = \delta = 0$, which contradicts our choice of λ . Hence either $\alpha = 0$ and $\beta = \pm 1$, in which case $\mathfrak{T}'(\mathbb{C}_K) = \mathbb{C}_K^\times \times \{1\}$, or $\beta = 0$ and $\alpha = \pm 1$, in which case $\mathfrak{T}'(\mathbb{C}_K) = \{1\} \times \mathbb{C}_K^\times$. \square

5.5. Let Λ' be the sublattice of Λ generated by $(q_1^c, 1)$ and $(1, q_2^c)$. Identify E_1 (resp. E_2) with $\mathbb{C}_K^\times / q_1^{c\mathbb{Z}}$ (resp. $\mathbb{C}_K^\times / q_2^{c\mathbb{Z}}$) in the obvious way. Let $A = E_1 \times E_2$, so $A(\mathbb{C}_K) = (\mathbb{C}_K^\times)^2 / \Lambda'$, and the kernel of the multiplication map $A \rightarrow J$ is $\Lambda / \Lambda' \cong (\mathbb{Z}/c\mathbb{Z})^2$. Since E_1 and E_2 are the only elliptic subvarieties of J , it follows that J is not isomorphic to a product of elliptic curves. Therefore the theta divisor of J is a smooth curve X of genus 2, and J is isomorphic to the Jacobian of X as principally polarized abelian varieties.

5.6. Since E_1 and E_2 are subvarieties of J , for $i = 1, 2$ the dual homomorphism $J \rightarrow E_i$ is an optimal quotient by (2.11). Let $\Gamma_1 = (q_1^c, 1)^{\mathbb{Z}}$ and $\Gamma_2 = (1, q_2^c)^{\mathbb{Z}}$, and for $i = 1, 2$ let Γ'_i be the saturation of Γ_i in Λ . Then $\Gamma'_1 = (q_1, \zeta)^{\mathbb{Z}}$ and $\Gamma'_2 = (\zeta, q_2)^{\mathbb{Z}}$. It follows from (4.17.2) that the cokernel of the map on component groups $\Phi_J \rightarrow \Phi_{E_i}$ is isomorphic to $\mathbb{Z}/c\mathbb{Z}$. In particular, $\Phi_J \rightarrow \Phi_{E_i}$ is not surjective. Note that the image of Λ in \mathbb{C}_K^\times under the evaluation map ev_{E_i} is generated by ζ and q_i – this is immediate from the definition of H in (5.2). This illustrates the surjectivity of the map $\Lambda \rightarrow c^{-1}\Gamma_i$ of (4.13.2).

5.7. A calculation involving p -adic Θ -functions shows that the Weil pairing on the c -torsion of the Tate curve E_i is given by the rule $e_c(\zeta, q_i) = \zeta$. Note that $\zeta \in E_i$ generates the subgroup of $E_i[c]$ which reduces to the identity component of the Néron model of E_i . Let $\psi : E_1[c] \rightarrow E_2[c]$ be the unique isomorphism such that $\psi(\zeta) = q_2$ and $\psi(q_1) = \zeta$. Then the graph

$$G = \{(P, \psi(P)) \mid P \in E_1[n]\}$$

is exactly the kernel of the map $A = E_1 \times E_2 \rightarrow J$, so this analytic construction coincides with the algebraic counterexample of Section 3, at least when $c = \text{ord}_K(q_1) = \text{ord}_K(q_2)$.

5.8. Let $E = E_1$. In the notation of Section 4, we have $q_E = q_1^c$, so $\text{ord}_K(q_E) = c \cdot \text{ord}_K(q_1)$. We can take $\lambda_E = (q_1, \zeta) \in \Lambda$, so

$$\langle \lambda_E, \lambda_E \rangle = \text{ord}_K H(q_1, \zeta)(q_1, \zeta) = \text{ord}_K(q_1).$$

It is clear that $m = \min\{\langle \lambda, \lambda_E \rangle > 0 \mid \lambda \in \Lambda\}$ is equal to $\langle \lambda_E, \lambda_E \rangle = \text{ord}_K(q_1) = \text{ord}_K(q_E)/c$. Hence $c\langle \lambda_E, \lambda_E \rangle = \text{ord}_K(q_E)$, so $c = n$ by (4.19.1), and hence $r = 1$ by (4.16.2). The fact that $r = 1$ is easy to see directly, as the idempotent e corresponds to the endomorphism $(a, b) \mapsto (a, 0)$ of the character group $\mathbb{Z}^2 \cong \Lambda$ of \mathfrak{T} , so $e \in \text{End}(\Lambda)$. The equality $n = c$ is then clear as well since the smallest power of the endomorphism $(x, y) \mapsto (x, 1)$ of $\mathbb{G}_{m,K}^2$ sending Λ to itself is c .

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